Three-point velocity correlation functions in two-dimensional forced turbulence

Denis Bernard*
Service de Physique Théorique de Saclay,† F-91191 Gif-sur-Yvette, France
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We present a simple exact formula for three-point velocity correlation functions in two-dimensional turbulence which is valid at all scales and which interpolates between the direct and inverse cascade regimes. As expected, these correlation functions are universal in these extreme regimes. We also discuss the effects of anisotropy and friction. [S1063-651X(99)06111-5]

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The aim of this paper is to set down an explicit formula for three-point velocity correlation functions in two-dimensional (2D) turbulence. See Eqs. (9) below and its consequences. This formula differs from the usual Kolmogorov formula, (cf. e.g., Ref. [1]), by the fact that it incorporates the existence of two inertial ranges which correspond to the inverse energy cascade and the direct enstrophy cascade, respectively. Although expected on scaling ground, this formula, and its simple proof, was surprisingly never spelled out in the turbulent literature. It is nevertheless one of the rare exact results on 2D turbulence. We thus feel that it was worth making it more public. This formula, and its large and short distance behaviors, are presented in Eqs. (14) and (16) below.

I. KRAICHNAN'S SCALING THEORY

A special feature which distinguishes two-dimensional from three-dimensional fluid mechanics is the conservation of vorticity moments in the inviscid limit. As first pointed out by Kraichnan in a remarkable paper [2], this opens the possibility for quite different scenarios for the behavior of turbulent flows in two and three dimensions. In two dimensions the inviscid Navier-Stokes equation admits two quadratic conserved quantities—the energy $\int(u^2/2)$ and the enstrophy $\int(\omega^2/2)$—with u the velocity and ω the vorticity. As argued by Kraichnan, if energy and enstrophy density are injected at a scale L_i, with respective rates $\bar{\epsilon}$ and $\bar{\epsilon}_w \simeq \bar{\epsilon} L_i^{-2}$, the turbulent system should react such that the energy flows toward large scales and enstrophy toward small scales. As this energy flow is opposite to the one involved in Kolmogorov's picture for 3D turbulence, one usually refers to the infrared energy flow as the inverse cascade and to the ultraviolet enstrophy flow as the direct cascade. The fact that energy has to escape to the large scales may be understood from the fact that in the absence of forcing the time variation of the energy is $\partial_t \int(u^2/2) = -\nu\int(\omega^2/2)$, with ν the viscosity. It thus vanishes in the inviscid limit $\nu \to 0$ if the enstrophy remains finite and the energy cannot be dissipated at small scales.

In the (IR) inverse cascade, scaling arguments lead to Kolmogorov's spectrum, with $E(k) \sim \bar{\epsilon}^{2/3} k^{-5/3}$ for the energy and $(\Delta u)(r) \sim (\bar{\epsilon} r)^{1/3}$ for the variation of the velocity on scale r. In the (UV) direct cascade, scaling arguments give Kraichnan's spectrum with $E(k) \sim \bar{\epsilon}_w^{2/3} k^{-3}$ for the energy and $(\Delta u)(r) \sim (\bar{\epsilon}_w r^3)^{1/3}$ for the velocity variation.

Of course the direct and inverse cascade have been extensively analyzed, both numerically, [see e.g., Ref. [3] and references therein for an (incomplete) sample of references], and theoretically (see e.g. Refs. [4,5] and references therein for a few relevent references, some of which discussing logarithmic corrections to Kraichnan's scaling). More recently, the inverse cascade has been observed experimentally, as described in Ref. [6]. Within experimental precision it shows no deviation from Kolmogorov's scaling.

II. MODEL AND ITS HYPOTHESIS

As usual, to model turbulent flows statistically, we consider the Navier-Stokes equation with an extra forcing term. Let $u^j(x,t)$ be the velocity field for an incompressible fluid, $\nabla \cdot u = 0$. In two dimensions the incompressibility implies that $u(x,t)$ derives from a stream function Φ such that $u^k = \epsilon_{kj}\partial_j \Phi$ with ϵ_{kj} the antisymmetric tensor. The Navier-Stockes equation reads

$$\partial_t u^j + (u\cdot\nabla)u^j - \nu\nabla^2 u^j = -\nabla^j p + f^j, \tag{1}$$

with p the pressure and $f(x,t)$ the external force such that $\nabla \cdot f = 0$. We choose the force to be Gaussian, white noise in time, with zero mean and two-point function:

$$\langle f^j(x,t) f^k(y,s)\rangle = C^{jk}(x-y) \quad \delta(t-s), \tag{2}$$

where $C^{jk}(x)$, with $\nabla^j C^{jk}(x) = 0$, is a smooth function varying on a scale L_i, quickly decreasing at infinity and regular at the origin. The scale L_i represents the injection length. We shall assume translation, rotation, and parity invariance, unless otherwise specified. Let $\hat{C}(x) \equiv \mathrm{tr}C(x)$. Its Taylor expansion at the origin will be denoted as $\hat{C}(x) = 2\bar{\epsilon} - \bar{\epsilon}_w r^2/2 + \cdots$, with $r^2 = x^k x^k$. The transversality condition $\nabla^j C^{jk}(x) = 0$ ensures that $\hat{C} = \nabla^k \Theta^k$ with $\Theta^k(x) = \bar{\epsilon} x^k - \bar{\epsilon}_w x^k r^2/8 + \cdots$ at short distances. A physical interpretation of $\bar{\epsilon}$ and $\bar{\epsilon}_w$ will be given later.

*Electronic address: dbernard@spht.saclay.cea.fr
†Laboratoire de la Direction des Sciences de la Matière du Commisariat à l'Energie Atomique.

The vorticity ω, with $\omega=\epsilon_{ij}\partial_i u_j$, is transported by the fluid:

$$\partial_t\omega+(u\cdot\nabla)\omega-\nu\nabla^2\omega=F, \tag{3}$$

with $F=\epsilon_{ij}\partial_i f_j$. The correlation function of the vorticity forcing term is thus

$$\langle F(x,t)F(y,s)\rangle=G(x-y)\delta(t-s) \tag{4}$$

with $G=-\nabla^2\hat{C}$. In particular, $G(0)=2\overline{\epsilon_w}$. The fact that the correlation function of the vorticity forcing is a gradiant will have physical consequences. Physically Eq. (3) means that for smooth solutions any power of the vorticity, and in particular the enstrophy $\int(\omega^2/2)$, is conserved in the absence of viscosity and forcing.

Since the inviscid limit is of course not under analytical control, we have to make a few hypotheses which encode Kraichnan's scenario of inverse and direct cascades. These hypotheses are the following: (i) the velocity correlation functions are assumed to be smooth at finite viscosity, and correlations of the velocity (without derivatives but at points coinciding or not) exist in the inviscid limit; (ii) Galilean invariant correlation functions, and in particular the velocity structure functions which are correlations of differences of the velocity field, are stationary; and (iii) in agreement with Kraichnan's picture, we demand that energy dissipative anomalies (but not enstrophy dissipative anomalies) be absent.

The two first hypotheses are standard in the statistical approach to turbulence, while the third is special to two dimensions. It follows by demanding that the enstrophy density $\Omega=\omega^2/2$ is finite in the inviscid limit, since the mean enstrophy density times the viscosity is equal to the mean dissipation rate, $\nu\langle\Omega\rangle=(\nu/2)\langle(\nabla u)\cdot(\nabla u)\rangle$.

III. VELOCITY CORRELATIONS

Let us look at the two-point velocity correlation function $\langle u(x)\cdot u(0)\rangle$. As is well known, (cf. e.g., Ref. [1]), it satisfies the following equation at finite viscosity:

$$\partial_t\langle u(x)u(0)\rangle-\tfrac{1}{2}\nabla_x^k\langle(\Delta u^k)(x)\ \ (\Delta u)^2(x)\rangle$$
$$+2\nu\langle(\nabla u)(x)\cdot(\nabla u)(0)\rangle=\hat{C}(x). \tag{5}$$

Here and in the following we shall denote velocity differences by $(\Delta u^k)(x)\equiv u^k(x)-u^k(0)$. Equation (5) assumes translation invariance, and uses the fact that the external force is Gaussian and white noise in time. Thanks to the fluid incompressibility the pressure drops out from this equation. The strategy consists of taking various limits of Eq. (5) in various orders. Let us take first the limit $x\to0$ followed by the inviscid limit. In this limit the second term in Eq. (5) vanishes due to the assumed smoothness of the correlation functions [hypothesis (i)]. Recall now hypothesis (iii) concerning the absence of energy dissipation. It in particular means that

$$\lim_{\nu\to0}\lim_{x\to0}\nu\langle(\nabla u)(x)\cdot(\nabla u)(0)\rangle=0. \tag{6}$$

Therefore, the third term in Eq. (5) also vanishes. This implies that in the inviscid limit the mean energy increases with time according to

$$\partial_t\left\langle\frac{u^2}{2}\right\rangle_{\nu=0}=\frac{1}{2}\hat{C}(0)=\overline{\epsilon}. \tag{7}$$

Thus $\langle u^2/2\rangle_{\nu=0}=\overline{\epsilon}t$ up to a constant, and $\overline{\epsilon}$ is indeed the energy injection rate. This is simply the obvious statement that in the absence of energy dissipation, and/or in the absence of friction or other processes by which the energy may escape, all energy injected into the system is transfered to the fluid. It is expected to be transfered to the mode with the smallest possible momentum, the so-called condensate [2]. In particular, Eq. (7) shows that in absence of energy dissipation a stationary state cannot be reached, although structure functions may converge at large times. This is one important difference between 2D and 3D turbulence.

Let us now assume that the two-point structure function is stationary, i.e., $\partial_t\langle(\Delta u)^2(x)\rangle=0$, [hypotheses (ii)]. From Eq. (5) one obtains, in the inviscid limit,

$$\tfrac{1}{2}\nabla_x^k\langle(\Delta u^k)(x)(\Delta u)^2(x)\rangle_{\nu=0}=2\overline{\epsilon}-\hat{C}(x). \tag{8}$$

Integrating this using parity invariance gives

$$\langle(\Delta u^k)(x)(\Delta u)^2(x)\rangle_{\nu=0}=2(\overline{\epsilon}x^k-\Theta^k(x)), \tag{9}$$

with $\nabla_x\cdot\Theta(x)=\hat{C}(x)$. Equation (9) together with Eq. (12) below fully determine the three point velocity correlation. Although simple to derive, this equation seems not to have appeared in the existing literature.

Equation (9) in particular shows that the inverse energy cascade takes place only if there is no dissipation anomaly, and thus only if the non-Galilean invariant velocity correlation functions do not reach a stationary state (in the absence of friction). Of course this is also a direct consequence of the physical fact that the energy condenses into the mode of smallest possible momemtum. As expected, Eq. (9) yields to Kolmogorov's scaling at large scale, since there $\boldsymbol{\Theta}^k(x)$ vanishes, and Kraichnan's scaling at small scale since $(\overline{\epsilon}x^k-\Theta^k(x))\sim r^3$ at short distance. But one can be a little more precise.

IV. VORTICITY CORRELATIONS

We now establish a formula for a mixed correlation function involving the vorticity and the velocity. Assuming that the structure functions of the velocity reach a stationary state implies that correlations of the vorticity also become stationary. The stationarity condition for the two-point vorticity functions, i.e., $\partial_t\langle\omega(x)\omega(y)\rangle=0$, implies

$$-\tfrac{1}{2}\nabla_x^k\langle(\Delta u^k)(x)\ \ (\Delta\omega)^2(x)\rangle+2\nu\langle\nabla\omega(x)\cdot\nabla\omega(0)\rangle$$
$$=G(x), \tag{10}$$

with $(\Delta\omega)(x)=\omega(x)-\omega(0)$. As for the velocity correlations, let us first take the limit of coincident point $x\to0$ at finite viscosity. The first term in Eq. (10) then vanishes by the hypotheses on the smoothness of the correlation functions at finite ν. Then taking the inviscid limit leads to

$$\lim_{\nu\to 0}\nu\langle(\nabla\omega)^2(x)\rangle=\tfrac{1}{2}G(0)=\overline{\epsilon_w}. \tag{11}$$

This is just the usual statement on enstrophy dissipative anomaly. It is equal to the enstrophy dissipation rate and the enstrophy injection rate. Let us now take the limits in the reversed order, the inviscid limit first. In that case the second term of Eq. (10) drops out, and one obtains

$$-\tfrac{1}{2}\nabla_x^k\langle(\Delta u^k)(x)(\Delta\omega)^2(x)\rangle_{\nu=0}=G(x).$$

Assuming isotropy and parity invariance, this gives

$$\langle(\Delta u^k)(x)(\Delta\omega)^2(x)\rangle_{\nu=0}=2\frac{x^k}{r}\left(\frac{d\hat{C}}{dr}\right)(r)$$
$$=-2\overline{\epsilon_w}x^k+O(r^2). \tag{12}$$

Thus this correlation is universal in the (UV) direct cascade regime. Its behavior and large scale depends on the way the forcing decreases at infinity. However, the fact that these correlations decrease faster than $O(1/r)$ at infinity is linked to the fact the vorticity forcing correlation is a gradiant. The ultraviolet behavior [Eq. (12)] was also described in Ref. [5]. This equation will be used to fix the coefficients of the infrared and ultraviolet expansions of the three-point velocity function left undetermined by Eq. (9). But Eq. (12) alone would not have been enough to determine these asymptotic expansions.

V. SCALING IN THE DIRECT AND INVERSE CASCADES

Let us first consider the short distance behavior in which the Kraichnan's direct cascade takes place. This corresponds to a scale much smaller than the injection length $x\ll L_i$. There, Taylor expanding Eq. (9) gives

$$\langle(\Delta u^k)(x)(\Delta u)^2(x)\rangle\simeq\tfrac{1}{4}\overline{\epsilon_w}x^k r^2.$$

Assuming isotropy and parity invariance, the three-point functions will be linear combinations of terms proportional to $x^i x^j x^k$ or $(\delta^{ij}x^k+\delta^{jk}x^i+\delta^{ki}x^j)r^2$. Among these two proportionality coefficients only one of them could be fixed using Eq. (9) only. However, the other coefficient is fixed by the exact result for correlation functions mixing the vorticity and the velocity, see Eq. (12). [One should use the relation $3\langle(\Delta u^z)(\Delta\omega)^2\rangle=4\partial_z^2\langle(\Delta u^z)^3\rangle$, where u^z is the velocity component in complex coordinates $z=x+iy$.] This then gives, for $x\to 0$,

$$\langle(\Delta u^i)(x)(\Delta u^j)(x)(\Delta u^k)(x)\rangle$$
$$\simeq\frac{\overline{\epsilon_w}}{8}((\delta^{ij}x^k+\delta^{jk}x^i+\delta^{ki}x^j)x^2-2x^ix^jx^k). \tag{13}$$

For the transverse and longitudinal correlations, this becomes

$$\langle(\Delta u)_\parallel^3\rangle=\langle(\Delta u)_\parallel(\Delta u)_\perp^2\rangle\simeq+\frac{\overline{\epsilon_w}}{8}\ r^3. \tag{14}$$

The coefficient $\overline{\epsilon_w}$ is equal to the mean enstrophy dissipation rate. Thus, as expected, the three-point velocity functions, which depend only on the enstrophy injection rate, are universal in the direct cascade. Equation (14) may be called the ''+1/8 law'' of the direct cascade.

Consider now the large distance behavior in which the Kraichnan's inverse cascade takes place, i.e., scale $x\gg L_i$. There, Taylor expanding Eq. (9) yields

$$\langle(\Delta u^k)(x)(\Delta u)^2(x)\rangle\simeq 2\overline{\epsilon}x^k.$$

There are two possible terms for the three-point functions, $(\delta^{ij}x^k+\delta^{jk}x^i+\delta^{ki}x^j)$ and $(x^ix^jx^k)/r^2$, whose coefficients cannot be completely fixed using only Eq. (9). But, again, these will be fixed by looking at the vorticity correlation functions [Eq. (12)]. One then obtains, for $x\to\infty$,

$$\langle(\Delta u^i)(x)(\Delta u^j)(x)(\Delta u^k)(x)\rangle\simeq\frac{\overline{\epsilon}}{2}(\delta^{ij}x^k+\delta^{jk}x^i+\delta^{ki}x^j), \tag{15}$$

with $\overline{\epsilon}$ the mean energy injection rate. Of course this gives the ''+3/2 law'' for the longitudinal statistics in the inverse cascade:

$$\langle(\Delta u)_\parallel^3\rangle=3\langle(\Delta u)_\parallel(\Delta u)_\perp^2\rangle\simeq+\frac{3\overline{\epsilon}}{2}r. \tag{16}$$

This law and Kolmogorov's scaling, as well as the existence of a condensate in which the energy accumulates were experimentally verified in Ref. [6].

VI. INFLUENCE OF ANISOTROPY AND PARITY SYMMETRY BREAKING

Anisotropy is irrelevent both in the ultraviolet and in the infrared. Indeed, let us model anisotropy by incorporating higher spin components in the forcing correlation function $C^{ij}(x)$, assuming that they are still regular at the origin and decrease at infinity. These components will be subdominant in Eq. (8) in both the ultraviolet (since the spin n component will behave as r^n) and the infrared (since they also vanish at infinity).

Suppose now that parity symmetry may be broken. Equation (9) is still valid (since it only assumes translation invariance) except that $\Theta^k(x)$ is determined up to $\Theta^k\to\Theta^k+\epsilon^{kj}\partial_j\Xi(r)$. This may change the ultraviolet scaling of the transverse velocity correlations but not the scaling of the longitudinal correlations $\langle(\Delta u)_\parallel^3\rangle$, although the amplitude may be modified.

VII. INFLUENCE OF FRICTION

In physical systems the infrared energy cascade will terminate at the largest possible scale at which the energy will escape. This could be mimicked by introducing a friction term in the Navier-Stokes equation, which then becomes

$$\partial_t u^j+(u\cdot\nabla)u^j-\nu\nabla^2u^j+\frac{1}{\tau}u^j=-\nabla^j p+f^j, \tag{17}$$

with τ the friction relaxation time, $\tau>0$. Friction brings another inviscid characteristic length L_f into the problem: $L_f \simeq \tau^{3/2}\overline{\epsilon}^{1/2}$. It increases as the friction is reduced, and one may suppose that $L_i \ll L_f$. This is the length at which the energy is extracted. The friction term dominates over the advection term at scales larger than L_f. So the direct cascade should take place at distances $x \ll L_i \ll L_f$, and the inverse cascade at distances $L_i \ll x \ll L_f$.

Under the same hypothesis as before, the mean energy density relaxes in the inviscid limit according to $\partial_t \langle u^2/2\rangle + (1/\tau)\langle u^2\rangle = \overline{\epsilon}$. It therefore reaches a stationary limit with $\langle u^2\rangle = \overline{\epsilon}\tau$. The stationarity of the two-point structure function, i.e., $\partial_t \langle (\Delta u)^2(x)\rangle = 0$, then gives, in the inviscid limit,

$$\frac{1}{2}\nabla_x^k \langle (\Delta u^k)(x)(\Delta u)^2(x)\rangle + \frac{1}{\tau}\langle (\Delta u)^2(x)\rangle = 2\overline{\epsilon} - \hat{C}(x). \tag{18}$$

Similarly, the stationarity of the vorticity correlations gives an equation similar to Eq. (10), but with an extra term representing the friction. As for the case without friction, let us first take the limit $x\to 0$ at finite viscosity, and then the inviscid limit. Let us denote the enstrophy injection rate by $\overline{\epsilon_w} = \frac{1}{2}G(0)$ and the enstrophy dissipation rate by $\hat{\epsilon}_w = \lim_{\nu\to 0} \nu \langle (\nabla\omega)^2(x)\rangle$. One then obtains $\langle \omega^2\rangle = \tau(\overline{\epsilon_w} - \hat{\epsilon}_w)$, with $\langle \omega^2\rangle = \lim_{\nu\to 0}\langle \omega^2(x)\rangle$. This simply means that the enstrophy density is equal to the difference of the enstrophy injection and the enstrophy dissipation rates times the friction relaxation time. In particular, if τ is finite, so is the enstrophy density. As a consequence the vorticity two-point correlation function $\langle \omega(x)\omega(0)\rangle$ will stay finite since it is bounded by $\langle \omega^2\rangle$. Taking the limit in reversed order, first $\nu\to 0$, yields the inviscid stationary equation:

$$-\frac{1}{2}\nabla_x^k \langle (\Delta u^k)(x)(\Delta\omega)^2(x)\rangle + \frac{2}{\tau}\langle \omega(x)\omega(0)\rangle = G(x). \tag{19}$$

Let us look at small distances in which the friction should be irrelevent. Let $\overline{\Omega} = \lim_{x\to 0}\langle \omega(x)\omega(0)\rangle$, which is expected to be equal to $\tau(\overline{\epsilon_w} - \hat{\epsilon}_w)$, although nothing prevents it from being different. Since $\overline{\Omega} < \tau\overline{\epsilon_w}$, the second term on the left-hand side of Eq. (19) cannot dominate, and $\nabla_x^k \langle (\Delta u^k)(x)(\Delta\omega)^2(x)\rangle \simeq \text{const}$ as $x\to 0$. This implies that the velocity three-point function $\langle (\Delta u)^3\rangle$ scale as r^3. In other words, Kraichnan's scaling of the three-point function is robust to friction in the direct cascade, although the amplitude may change.

More precisely, suppose that $\overline{\Omega}$ is finite and nonvanishing. Then the scaling formula (13) for the three-point function still holds, but with $\overline{\epsilon_w}$ replaced by $(\overline{\epsilon_w}\tau - \overline{\Omega})/\tau$. Recall that it is likely that $(\overline{\epsilon_w}\tau - \overline{\Omega})/\tau$ is equal to the enstrophy dissipation rate $\hat{\epsilon}_w$, meaning that in the presence of friction one simply has to replace the injection rate by the dissipation rate in formula (13). Moreover, the finiteness of the vorticity two-point function at coincident points also implies that $\langle (\Delta u)^2(x)\rangle \simeq \overline{\Omega}/2r^2$, since $\nabla_x^2 \langle (\Delta u)^2(x)\rangle = 2\langle \omega(x)\omega(0)\rangle$. This scaling may be broken only if $\overline{\Omega} = \lim_{x\to 0}\langle \omega(x)\omega(0)\rangle_{\nu=0}$ vanishes. It is worth specifying in which scale domain this behavior will be valid. At finite viscosity, there are various ultraviolet characteristic lengths; the usual dissipative lengths $r_d \simeq \nu^{3/4}\epsilon^{1/4}$ and $l_d \simeq \nu^{1/2}\overline{\epsilon_w}^{-1/6}$, and another friction length $l_f \simeq \nu^{1/2}\tau^{1/2}$ above which friction dominates over dissipation. Since $r_d \ll l_d \simeq l_f$ in the limit we are considering, $\nu\to 0$ and τ fixed, this scaling will be valid for $l_d \simeq l_f \ll x \ll L_i$.

Let us now consider distances larger than the injection length L_i but smaller than the friction length L_f. Then $2\overline{\epsilon} - \hat{C}(x) \simeq 2\overline{\epsilon}$, and the positivity argument cannot be applied. However, unless miraculous cancellations occur between the two terms on the left-hand side of Eq. (18) (which would mean that the domains in which advection or friction dominate intertwine), the correlation $\langle (\Delta u^k)(x)(\Delta u)^2(x)\rangle$ will still scale as r. Clearly this argument is less robust than the one used for the short distance analysis.

VIII. CONCLUSIONS

Besides giving the expected formula for the three-point velocity correlation functions, this short proof also indicates that if the inverse cascade takes place, as experimentally verified, then, in absence of friction, the non-Galilean invariant velocity correlation functions do not become stationary, although structure functions do. However, it gives no hints on how to decipher the behavior of the vorticity, one of the main challanging problems of two dimensional turbulence.

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